Hardy-Hilbert不等式一个新的改进

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摘要: 研究关于重级数型 Hardy-Hilbert不等式改进的问题。通过给出如下形式的权系数的估计式

$$\omega(q,n)=\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\frac{1}{2n^{\frac{1}{p}}+\frac{2}{e}n^{-\frac{1}{q}}},\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\ n\in\mathbf{N},$$

从而得到 Hardy-Hilbert不等式的一个新的改进形式

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\frac{1}{2n^{\frac{1}{p}}+\frac{2}{e}n^{-\frac{1}{q}}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\frac{1}{2n^{\frac{1}{q}}+\frac{2}{e}n^{-\frac{1}{p}}}\right]b_n^q\right\}^{\frac{1}{q}}。$$

关键词: Hardy-Hilbert不等式; 重级数; 权系数; Hölder不等式
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A New Improvement of Hardy-Hilbert Inequality

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Abstract: This paper deals with the refinement of the Hardy-Hilbert inequality for double series. A new inequality for the weight coefficient $\omega(q,n)$ in the form is proved

$$\omega(q,n)=\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\frac{1}{2n^{\frac{1}{p}}+\frac{2}{e}n^{-\frac{1}{q}}},\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\ n\in\mathbf{N},$$

and a new generalization and improvement of Hardy-Hilbert inequality is obtained.
Key words: Hardy-Hilbert inequality; double series; weight coefficient; Hölder inequality

0 引言

设 $\{a_n\}$和 $\{b_n\}$是两个非负实数序列, $\frac{1}{p}+\frac{1}{q}=1$且 $p>1$, 若 $\sum_{n=1}^{\infty}a_n^p<+\infty$且 $\sum_{n=1}^{\infty}b_n^q<+\infty$, 那么

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}。\tag{1}$$

这是有名的 Hardy-Hilbert重级数定理[1]。近年来, 不等式(1)有各种推广和改进(见文献[2-7])。本文的目的是要进一步改进文献[6-7]的结果, 采用对给出的权系数进行估计的方法。

1 引理及其证明

引理 1[7] 设 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $n\in\mathbf{N}$, 则

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$$\omega(q,n) < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{f_n(p) + g_n(p)}{n^{\frac{1}{p}}}。 \quad (2)$$

这里 $\omega(q,n)$同摘要中的 $\omega(q,n)$，且

$$f_n(p) = p + \frac{1}{12p} + \frac{1}{(1+p)n} + \frac{1}{12pn^2} + \frac{1}{3(1+3p)n^3}, \quad (3)$$

$$g_n(p) = \frac{-1}{12pn} - \frac{1}{2(1+2p)n^2} - \frac{7}{12} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{7}{12n^3}。 \quad (4)$$

引理 2[6] 设 $f(x)$为$[1,+\infty)$的四阶连续可微函数，若 $f^{(4)}(x) \geqslant 0$，$f'''(1) < 0$，$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} f^{(i)}(x) = 0$，$(i=1,2,3)$且$\sum\limits_{m=1}^{\infty} f(m)$及$\int_1^{\infty} f(x)\mathrm{d}x$收敛，则

$$\sum_{m=1}^{\infty} f(m) \leqslant \int_1^{\infty} f(x)\mathrm{d}x + \frac{1}{2}f(1) - \frac{1}{12}f'(1)。 \quad (5)$$

引理 3 设 $p>1$，$n\in\mathbf{N}$，其中 $f_n(p)$，$g_n(p)$同引理 1中定义，则

$$f_n(p) + g_n(p) > \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}。 \quad (6)$$

证明

$$f'_n(p) = 1 - \frac{1+n^2}{12n^2p^2} - \frac{1}{(1+p)^2n} - \frac{1}{(1+3p)^2n^2} >$$
$$1 - \frac{1+n^2}{12n^2} - \frac{1}{(1+1)^2n} - \frac{1}{(1+3)^2n^2} =$$
$$\frac{11}{12} - \frac{1}{12n^2} - \frac{1}{4n} - \frac{1}{16n^3} > 0,$$

且

$$g'_n(p) = \frac{1}{12p^2n} + \frac{1}{(1+2p)^2n^2} > 0。$$

所以 $f_n(p)+g_n(p)$在$(1,+\infty)$上是增函数，则

$$f_n(p) + g_n(p) > \lim(f_n(p) + g_n(p)) = \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}。$$

证毕。

引理 4 设 $q>1$，$\frac{1}{p}+\frac{1}{q}=1$，$n\in\mathbf{N}$，则

$$\omega(q,n) < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{p}} + \frac{2}{e}n^{-\frac{1}{q}}}, \quad (7)$$

$$\omega(p,n) < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{q}} + \frac{2}{e}n^{-\frac{1}{p}}}。 \quad (8)$$

证明 当 $n\geqslant 3$时，

$$\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right)\left(1 + \frac{1}{en}\right) = \frac{1}{2} + \frac{1}{n}\left(\frac{1}{2e} - \frac{1}{12} - \frac{1}{12en} - \frac{1}{2n^2} - \frac{1}{2en^3}\right), \quad (9)$$

$$\frac{1}{2e} - \frac{1}{12} - \frac{1}{12en} - \frac{1}{2n^2} - \frac{1}{2en^3} = \frac{6n^3 - en^3 - n^2 - 6en - 6}{12en^3} > 0。 \quad (10)$$

事实上，(10)式等价于当 $n\geqslant 3$时，

$$\Phi(n) = 6n^3 - en^3 - n^2 - 6en - 6 > 0,$$

由 $\Phi(3) = 147 - 45e > 0$，又

$$\Phi'(n) = 18n^2 - 3en^2 2n - 6e,$$

$$\Phi'(3) = 156 - 33e > 0,$$
$$\Phi''(n) = 36n - 6en - 2,$$
$$\Phi''(3) = 106 - 18e > 0,$$
$$\Phi'''(n) = 36 - 6e > 0,$$

从而当 $n \geqslant 3$时, $\Phi''(n) > 0$, $\Phi'(n) > 0$, $\Phi(n) > 0$。因此由(10)式有

$$\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right)\left(1 + \frac{1}{en}\right) > \frac{1}{2},$$

即

$$\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3} > \frac{1}{2 + \frac{2}{e}n^{-1}}。\tag{11}$$

由引理 1、引理 3得

$$\omega(q, n) < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{p}} + \frac{2}{e}n^{-\frac{1}{q}}}, n \geqslant 3。\tag{12}$$

当 $n = 1$时, 由文献[6]有

$$\omega(q, 1) < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1 - C}{1} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2 + \frac{2}{e}}。\tag{13}$$

其中: C 是欧拉常数, 且 $1 - C \approx 0.422\,784\,33\cdots$。

当 $n = 2$时, 由 $C < 0.577\,21$, 得

$$\frac{1}{2\left(2^{\frac{1}{p}}\right) + \frac{2}{e}\left(2^{-\frac{1}{q}}\right)} < \frac{1 - C}{2^{\frac{1}{p}}},$$

所以

$$\omega(q, 2) < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1 - C}{2^{\frac{1}{p}}} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2\left(2^{\frac{1}{p}}\right) + \frac{2}{e}\left(2^{-\frac{1}{q}}\right)}。\tag{14}$$

由(12)~(14)式得(7)式成立, 换 q为 p, 加之$\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} = \frac{\pi}{\sin\left(\frac{\pi}{q}\right)}$, 得(8)式成立。

2　主要结果

定理　若 $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geqslant 0$, $b_n \geqslant 0$, $\sum_{n=1}^{\infty} a_n^p < +\infty$, $\sum_{n=1}^{\infty} b_n^q < +\infty$, 则摘要中的重级数不等式成立, 且

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m + n}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{p-1}\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{p}} + \frac{2}{e}n^{-\frac{1}{q}}}\right]a_n^p。\tag{15}$$

证明　由 Hölder不等式, 有

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m + n} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{(m + n)^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}a_m\right]\left[\frac{1}{(m + n)^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}b_n\right] \leqslant \\
&\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m + n}\left(\frac{m}{n}\right)^{\frac{1}{q}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m + n}\left(\frac{n}{m}\right)^{\frac{1}{p}}b_n^q\right\}^{\frac{1}{q}} = \\
&\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{1}{m + n}\left(\frac{n}{m}\right)^{\frac{1}{q}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{1}{m + n}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right]b_n^q\right\}^{\frac{1}{q}} = \\
&\left\{\sum_{n=1}^{\infty}\omega(q, n)a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(p, n)b_n^q\right\}^{\frac{1}{q}}。
\end{aligned}$$

因此, 将(7)、(8)式代入上式即得结论成立。

由 $\omega(p,n) < \dfrac{\pi}{\sin\left(\dfrac{\pi}{p}\right)}$, 再由 Hölder不等式, 得

$$\sum_{n=1}^{\infty}\frac{a_n}{m+n} = \sum_{n=1}^{\infty}\left[\frac{a_n}{(m+n)^{\frac{1}{p}}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}\right]\left[\frac{1}{(m+n)^{\frac{1}{q}}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}\right] \leqslant$$

$$\left\{\sum_{n=1}^{\infty}\left[\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{p}}\right\}^{\frac{1}{q}} =$$

$$\left\{\sum_{n=1}^{\infty}\left[\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}\right]a_n^p\right\}^{\frac{1}{p}}\{\omega(p,n)\}^{\frac{1}{q}} <$$

$$\left\{\sum_{n=1}^{\infty}\left[\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}\right]a_n^p\right\}^{\frac{1}{p}}\left\{\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right\}^{\frac{1}{q}}。$$

由(7)式有

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^p < \left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{\frac{p}{q}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}a_n^p =$$

$$\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{p-1}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{q}}\right]a_n^p =$$

$$\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{p-1}\sum_{n=1}^{\infty}\omega(q,n)a_n^p <$$

$$\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^{p-1}\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{p}}+\frac{2}{e}n^{-\frac{1}{q}}}\right]a_n^p。$$

所以(15)式成立。定理证毕。

显然

$$\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{p}}+\frac{2}{e}n^{-\frac{1}{q}}} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{1}{2n^{\frac{1}{p}}+n^{-\frac{1}{q}}},$$

$$q > 1, \frac{1}{p}+\frac{1}{q} = 1, n \in \mathbf{N}。 \tag{16}$$

所以, 上面定理所得结论比文献[7]的最终结论更加精确。

参考文献:

[1] Hardy G H, Littlewood J E, Polya G. Inequalities[M]. Cambridge: Cambridge University Press, 1952.
[2] Gao Mingzhe, Yang Bicheng. On the extended Hilbert's inequality[J]. Proc Amer Math Soc, 1998, 126(3): 751-759.
[3] Gao Mingzhe. A note on the Hardy-Hilbert inequality[J]. J Math Anal Appl, 1996, 204(1): 346-351.
[4] Yang Bicheng. On an extension of Hardy-Hilbert's inequality[J]. Chinese Ann Math, 2002, 23(2): 247-254.
[5] Xu L C, Gau Y K. Note on Hardy-Riesz' extension of Hilbert's inequality[J]. Chinese Quarterly Journal of Mathematics, 1991, 6(1): 75-77.
[6] 杨必成,高明哲. 关于 Hardy-Hilbert不等式的一个最佳常数[J]. 数学进展, 1997, 26(2): 159-164.
[7] Yang Bicheng, Debnath L. On a new strengthened Hardy-Hilbert's inequality[J]. Int J Math Sci, 1998, 21(2): 403-408.
[8] 杨必成. 关于一般 Hilbert二重级数定理的改进[J]. 数学研究, 1996, 29(2): 64-70.